Global Conformal Invariance and Bilocal Fields with Rational Correlation Functions*

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The singular part of the *operator product expansion* (OPE) of a pair of *globally conformal invariant* (GCI) scalar fields ϕ of (integer) dimension d can be written as a sum of the 2–point function of ϕ and $d-1$ bilocal conformal fields $V_\nu\,(x_1,\, x_2)$ of dimension $(\nu,\, \nu)\,,\ \nu = 1,\, \dots,\, d-1\,.$ As the correlation functions of $\phi\,(x)$ are proven to be rational [6], we argue that the correlation functions of V_ν can also be assumed rational. Each $V_\nu\,(x_1,\, x_2)$ is expanded into local symmetric tensor fields of *twist* (dimension minus rank) $2\nu\,.$ The case $d = 2\,,$ considered previously [5], is briefly reviewed and current work on the $d = 4$ case (of a Lagrangean density in 4 space–time dimensions) is previewed.

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1 Introduction

Our study [5] of the theory of a GCI hermitean scalar field of dimension 2 suggests the following generalization.

Given a GCI neutral scalar field ϕ of dimension $d\,(\in\, \mathbb{N})$ in 4 dimensional Minkowski space we look for an OPE of the product of two ϕ's in *bilocal fields*:

$$\phi\,(x_1)\;\phi\,(x_2) \;=\; \langle 12\,\rangle \,+\, \sum_{\nu\,=\,1}^{d-1}\,(12)^{d-\nu}\;V_\nu\,(x_1,\,x_2) \,+\, :\!\phi\,(x_1)\,\phi\,(x_2)\!:\ . \tag{1}$$

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Here we are using the following shorthand notation of [5]:

$$\begin{aligned}
\langle 1 \ldots n \rangle \; &= \; \langle 0 | \, \phi \, (x_1) \ldots \phi \, (x_n) \, | 0 \rangle \; , \\
\langle 12 \rangle \; &= \; N_\phi \, (12)^d \; , \quad (12) \; = \; \frac{1}{4\pi^2 \rho_{12}} \, , \quad \rho_{12} \; = \; x_{12}^{\,2} + i \, 0 \, x_{12}^0
\end{aligned} \tag{2}$$

(the metric signature is space–like: $x^2 := \mathbf{x}^2 - x_0^2$, $\mathbf{x}^2 = x_1^2 + x_2^2 + x_3^2$.) The bilocal conformal field $V_\nu \, (x_2, \, x_2)$ of dimension $(\nu, \, \nu)$ can be expanded in a series of twist 2ν local symmetric traceless tensor fields. Each term in this expansion is universal, only the (numerical) coefficients depend on the theory – i. e., on the dimension d and (possible additional assumptions on) the field ϕ. As a consequence, the fields V_ν and the *normal product*: $: \phi \, (x_1) \, \phi \, (x_2) :$ defined by (1) are mutually orthogonal:

$$\begin{aligned}
&\langle 0 | \, V_\nu \, (x_1, \, x_2) \, | 0 \rangle \; = \; 0 \; = \; \langle 0 | \, V_\lambda \, (x_1, \, x_2) \; V_\nu \, (x_3, \, x_4) \, | 0 \rangle \; , \; \text{for} \quad \lambda \neq \nu \, , \\
&\langle 0 | \, V_\nu \, (x_1, \, x_2) \, : \phi \, (x_3) \, \phi \, (x_4) : \, | 0 \rangle \; = \; 0 \, , \quad \lambda, \, \nu \; = \; 1, \, \ldots, \, d-1 \, .
\end{aligned} \tag{3}$$

Let s_{ij} be the substitution exchanging the arguments x_i and x_j of a function of several 4–vectors. Then the symmetrized contribution F_ν of twist 2ν tensor fields to the truncated 4–point function:

$$\begin{aligned}
\mathcal{W}_4^t \, (d) \; &\equiv \; \mathcal{W}^{\, t} \, (x_1, \, x_2, \, x_3, \, x_4; \, d) \; := \\
&= \; \langle 1234 \rangle - \langle 12 \rangle \, \langle 34 \rangle - \langle 13 \rangle \, \langle 24 \rangle - \langle 14 \rangle \, \langle 23 \rangle \; ,
\end{aligned} \tag{4}$$

$$\begin{aligned}
F_\nu \, (x_{12}, \, x_{23}, \, x_{34}; \, d) \; = \; & (1 + s_{23} + s_{13}) \, (12)^{d-\nu} \, (34)^{d-\nu} \; \times \\
& \times \; \langle 0 | \, V_\nu \, (x_1, \, x_2) \; V_\nu \, (x_3, \, x_4) \, | 0 \rangle \; ,
\end{aligned} \tag{5}$$

is a crossing symmetric rational function of ρ_{ij} (2). The vacuum expectation value of the product of two V_1 can be written as

$$\begin{aligned}
\langle 0 | \, V_1 \, (x_1, \, x_2) \; V_1 \, (x_3, \, x_4) \, | 0 \rangle \; &= \; [(13) \, (24) + (14) \, (23)] \, f \, (s, \, t) \; = \\
&= \; (13) \, (24) \left(1 + t^{-1} \right) \, f \, (s, \, t) \; ,
\end{aligned} \tag{6}$$

where $f \, (= f_d)$ is a function of the conformally invariant cross ratios

$$s \; = \; \frac{\rho_{12} \rho_{34}}{\rho_{13} \rho_{24}} \, , \quad t \; = \; \frac{\rho_{14} \rho_{23}}{\rho_{13} \rho_{24}} \, , \tag{7}$$

satisfying the s_{12}–symmetry condition

$$(s_{12} f) \, (s, \, t) \; := \; f \left(\frac{s}{t}, \, \frac{1}{t} \right) \; = \; f \, (s, \, t) \; . \tag{8}$$

The present paper is devoted to a general study of the field $V_1(x_1, x_2)$ which involves the (even rank) conserved symmetric traceless tensors

$$T_l(x, \zeta) = T(x)_{\mu_1 \dots \mu_l} \zeta^{\mu_1} \dots \zeta^{\mu_l}, \quad \Box_\zeta T_l(x, \zeta) = 0 = \frac{\partial^2}{\partial x^\mu \partial \zeta_\mu} T_l(x, \zeta) \tag{9}$$

in its expansion in local fields. The simplification in this case stems from the fact that all 3–point functions $\langle 0| V_1(x_1, x_2)\, T_{2l}(x_3, \zeta) |0\rangle$, whose expressions are derived from conformal invariance alone, satisfy the d'Alembert equation in both x_1 and x_2; as a result, so does V_1:

$$\Box_1 V_1(x_1, x_2) = 0 = \Box_2 V_1(x_1, x_2), \quad \Box_j = \frac{\partial^2}{\partial x_j^\mu \partial x_{j\,\mu}}, \quad j = 1, 2. \tag{10}$$

This allows to compute the function $f_d(s, t)$ (of Eq. (6)) in terms of $d-1$ (real) constants. For $d = 2$ the commutator algebra generated by $V_1(x_1, x_2)$ is relatively simple: it coincides with a central extension of the infinite (real) symmetric Lie algebra $\widehat{sp}(\infty)$ [5].

GCI allows to formulate the theory in compactified Minkowski space $\overline{M}$, which admits a convenient realization as a $U(1)$ bundle over $\mathbb{S}^3$:

$$\overline{M} = \frac{\mathbb{S}^3 \times \mathbb{S}^1}{\mathbb{Z}_2} = \Big\{ z_\alpha = e^{2\pi i \vartheta} u_\alpha ; \ \vartheta \in \mathbb{R}/\mathbb{Z}, \ (u_\mu) = (\mathbf{u}, u_4) \in \mathbb{S}^3 \ (\text{i.e. } u^2 = \mathbf{u}^2 + u_4^2 = 1) \Big\}, \tag{11}$$

single valued (observable) fields being periodic of period 1 with respect to the conformal time variable ϑ (moreover, z_α does not change for $\vartheta \mapsto \vartheta + \frac{1}{2}$, $u_\alpha \mapsto -u_\alpha$). The passage from the real Minkowski space coordinates x_μ to the complex $\overline{M}$ coordinates z_α is given by the complex conformal transformation

$$\mathbf{z} = \frac{\mathbf{x}}{\omega(x)}, \ z_4 = \frac{1 - x^2}{2\,\omega(x)}, \ \omega(x) = \frac{1 + x^2}{2} - i\, x^0 \ \left(z^2 := \mathbf{z}^2 + z_4^2 = \frac{\overline{\omega(x)}}{\omega(x)} \right) \tag{12}$$

and by an accompanying field transformation (see [9] or Sec. 4 of [5]). The compact picture fields have a natural decomposition into discrete modes. The presence of such a discrete basis simplifies the study of the unitarity condition for the vacuum representation of $\widehat{sp}(\infty)$ [5] reviewed in Sec. 2.

Sec. 3 is devoted to the study of the contribution of V_1 to the truncated 4–point function $\mathcal{W}_4^t$ in the case of a field $\mathcal{L}$ of dimension 4 with the properties of a (gauge invariant) Lagrangean density. We demonstrate that this contribution can only be recovered by the Lagrangean $\mathcal{L}_0$ of a *free* (Maxwell) field for a special choice ($a_1 = a_2$) of the two parameters involved (after having excluded the presence of a $d = 2$ scalar field). In Sec. 4 we construct the crossing symmetrized contribution of the twist 4 tensor fields to $\mathcal{W}_4^t$ which involves two more parameters. These results indicate that there is room for a nontrivial GCI theory of a $d = 4$ field with rational correlation functions.

2 OPE for $V_1(x_1, x_2)$. Infinite symplectic algebra for $d = 2$ and Hilbert space positivity

The story of conformally invariant OPE goes back over 30 years – see [3]; for a sample of later reviews and further developments we refer to [1, 10, 7, 4, 2]. Here we summarize and extend to an arbitrary ϕ the results of Sec. 3 and Appendix A of [5].

For any dimension d of the basic field ϕ the expansion of V_1 into local tensor fields of type (9) assumes the form

$$V_1(x_1, x_2) = \sum_{l=0}^{\infty} C_l(d) \int_0^1 d\alpha\, K_l(\alpha, \rho_{12}\,\Box_2)\, T_{2l}(x_2 + \alpha x_{12}, x_{12}) \,, \tag{13}$$

where

$$K_l(\alpha, z) = \frac{(4\,l+1)!}{[(2\,l)!]^2}\, \alpha^{2l}\, (1-\alpha)^{2l} \sum_{n=0}^{\infty} \frac{\left[\alpha(\alpha-1)\,\frac{z}{4}\right]^n}{n!\,(4l+1)_n}\,, \quad (\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)}\,, \tag{14}$$

$\Box_2$ is the d'Alembert operator acting on x_2 for fixed x_{12}. The integro-differential operator in (13) transforms the 2–point function of T_{2l} into a 3–point one (see [2]); for light–like auxiliary variable ζ this relation assumes the form:

$$\int_0^1 d\alpha\, K_l(\alpha, \rho_{12}\,\Box_2)\, \frac{(x_{12}\cdot r(y(\alpha))\cdot\zeta)^{2l}}{\rho_{y(\alpha)}^{2l+2}} = \frac{(X\cdot\zeta)^{2l}}{\rho_{13}\,\rho_{23}}\,, \text{ with } y(\alpha) := x_{23} + \alpha\, x_{12}\,; \tag{15}$$

here

$$X = X_{12}^3 := \frac{x_{13}}{\rho_{13}} - \frac{x_{23}}{\rho_{23}}\,, \quad \zeta_1\cdot r(y)\cdot\zeta_2 = \zeta_1\cdot\zeta_2 - 2\,\frac{(\zeta_1\cdot y)\,(\zeta_2\cdot y)}{\rho_y}\,, \tag{16}$$

$$\rho_{y(\alpha)} = y^2(\alpha) + i\,0\,y(\alpha)^0 = \alpha\,\rho_{13} + (1-\alpha)\,\rho_{23} - \alpha(1-\alpha)\,\rho_{12}\,. \tag{17}$$

(Had we started with a complex scalar field ϕ and with the OPE of $\phi^*(x_1)\,\phi(x_2)$ instead of (1) we would have also encountered odd rank symmetric tensors $T_{2l+1}(x,\,x_{12})$ in the expansion of $V_1(x_1,\,x_2)$.)

The contribution of T_{2l} to the 4–point function (6) is universal (up to normalization) and is expressed in terms of hypergeometric functions (see Eq. (3.10) of [2] or (A.6) of [5]). It is determined by its value on the light cone $\rho_{34} \,=\, 0\,(=\,s\,)$ (cf. Appendix A of [5]) for which its expression is particularly simple:

$$\langle 0|\ V_1(x_1,\,x_2)\int_0^1 d\alpha\,\frac{\alpha^{2l}(1-\alpha)^{2l}}{B(2l+1,2l+1)}\ T_{2l}(x_4+\alpha x_{34},\,x_{34})\,|0\rangle\ =$$
$$=\ A_l\ (13)\,(24)\,(1-t)^{2l}\ F\,(2l+1,\,2l+1;\,4l+2;\,1-t)$$
$$(\,B\,(\mu,\,\nu)\ :=\ \int_0^1 x^{\mu-1}\,(1-x)^{\nu-1}\ dx\ =\ \frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}\,)\,. \tag{18}$$

The constant A_l (proportional to $C_l\,(d)$) provides the normalization of the 3–point function

$$\langle 0|\ V_1(x_1,\,x_2)\ T_{2l}(x_3,\,\zeta)\,|0\rangle\ =\ A_l\,(12)\,(X^{\,2})^{l+1}\,(\zeta^{\,2})^l\ C^1_{2l}\left(\widehat{\zeta}\cdot\widehat{X}\right)$$
$$(=\ \frac{A_l}{(2\,\pi)^2}\ \frac{(2\,\zeta\cdot X)^{2l}}{\rho_{13}\,\rho_{23}}\ \ \text{for}\ \zeta^{\,2}\ =\ 0\,)\,,\quad \widehat{v}\ =\ \frac{v}{\sqrt{v^{\,2}}}\quad (\,A_l\ =\ N_l\,C_l\,) \tag{19}$$

($N_l > 0$ fixing the normalization of the 2–point function of T_{2l}). The expansion parameter $1-t$ tends to zero whenever the light–like vector x_{34} does:

$$1-t\ =\ 2\left(\frac{x_{24}}{\rho_{24}}\ -\ \frac{x_{13}}{\rho_{13}}\right)\cdot x_{34} + 4\,\frac{(x_{13}\cdot x_{34})\,(x_{24}\cdot x_{34})}{\rho_{13}\,\rho_{24}}\quad \text{for}\quad \rho_{34}\ =\ 0\,. \tag{20}$$

If we change the normalization of T_{2l} setting $T_{2l} \mapsto Z_l\,T_{2l}$ (keeping ϕ fixed) then the constants $C_l\,(d)$ and A_l appearing in (13) and (19) will also change, $A_l \mapsto Z_l\,A_l$, $C_l \mapsto Z_l^{-1}\,C_l$ ($N_l \mapsto Z_l^2\,N_l$), but their product $A_l\,C_l$ will remain the same and it can be determined by the 4–point function (6). We, therefore, proceed to compute the general form of this 4–point function.

According to (10) it satisfy the d'Alembert equation in x_j . This implies a second order partial differential equation for $f_d\,(s,\,t)$:

$$\Big\{\left(1+t^2-s\right)D_s + s\,(t-1)\,D_t + (1+t)\left[t\,D_s^2 + s\,D_t^2 +\right.$$
$$\left. +\ (s+t-1)\,D_s\,D_t\right]\Big\}\ f_d\,(s,\,t)\ =\ 0\,,\quad D_u\ =\ u\frac{\partial}{\partial u}\quad (\,u\ =\ s,\,t\,)\,. \tag{21}$$

We are looking for a solution of (21) for which the product $t^{d-2}\,(1+t)\,\times$ $f_d\,(s,\,t)$ is a polynomial in s and t of overall degree not exceeding $2d-3$. (This requirement follows from Wightman positivity [8]: for space–time dimensions $D\,>\,2$, the truncated 4–point function has a strictly smaller singularity for $\rho_{ij}\to 0$ $(1\leq i<j\leq 4)$ than the corresponding 2–point function – see [6, 5].) Such a solution can be obtained starting with an "initial condition" in s that obeys the symmetry property (8):

$$f_d\,(0,\,t)\;=\;\sum_{\nu\,=\,0}^{d-2}\,a_\nu\,\frac{(1-t)^{2\nu}}{t^\nu}\quad\left(\,=\;f_d\left(0,\,\frac{1}{t}\right)\,\right). \tag{22}$$

In particular, for $d\,=\,2$, $f_2\,(s,\,t)$ is a constant (which we shall denote by c since it is analogous to the Virasoro central charge – [5]). In this case the products $B_l := A_l\,C_l\,(2)$ can be calculated from the seemingly overdetermined infinite set of algebraic equations stemming from

$$\begin{aligned} c\left(1+\frac{1}{t}\right)\ \left(\,=\;c\left\{2+\sum_{n\,=\,1}^{\infty}\,(1-t)^n\right\}\,\right)\;=\;\\ =\;\sum_{l\,=\,0}^{\infty}\,B_l\,(1-t)^{2l}\,F\,(2l+1,\,2l+1;\,4l+2;\,1-t)\;. \end{aligned} \tag{23}$$

The result is ([5])

$$B_l\;\left(\,=\;A_l\,C_l\,(2)\;=\;N_l\,C_l^2\,(2)\,\right)\;=\;\frac{2\,c}{\binom{4\,l}{2\,l}}\qquad\left(\,\text{i.e.,}\;\;B_0\;=\;2\,c\,,\;\;B_1\;=\;\frac{c}{3}\quad\text{etc.}\,\right). \tag{24}$$

We see that for $c\,>\,0$ all expansion coefficients (24) are positive thus *a necessary condition for Hilbert space* (*Wightman*) *positivity* is satisfied. This condition is not, however, sufficient. We shall briefly review the argument of [5] which establishes a necessary and sufficient positivity condition.

One first observes that for $c\,=\,N\,(\in\,\mathbb{N}\,)$ the OPE algebra of the bilocal field $V_1\,(x_1,\,x_2)$ coincides (for $d\,=\,2$) with the algebra of a sum of normal products of mutually commuting free massless scalar fields φ_i:

$$V_1\,(x_1,\,x_2)\;=\;\sum_{i\,=\,1}^{N}\,:\!\varphi_i\,(x_1)\,\varphi_i\,(x_2)\!:\quad\text{for}\quad c\;=\;N\;(\,=\;1,\,2,\,\ldots\,)\,, \tag{25}$$

where

$$\langle 0|\;\varphi_i\,(x_1)\,\varphi_j\,(x_2)\,|0\rangle\;=\;\delta_{ij}\;(12\,)\;\left(\,=\;\delta_{ij}\;\frac{1}{4\,\pi^2\,\rho_{12}}\;\right). \tag{26}$$

Since each φ_i generates under commutation an (infinite) Heisenberg (Lie) algebra, it follows that V_1 generates a central extension of the infinite (real) symplectic algebra.

To find a stronger positivity condition for the vacuum representation of V_1 we shall, following Sec. 5 of [5], use the compact picture field $V_1(z_1, z_2)$ (expressed in terms of the coordinates (11)). To this end we observe that in accord with a general theorem of Borchers the restriction of V_1 to a 1–dimensional bilocal field $v(\mathfrak{z}_1, \mathfrak{z}_2)$ for fixed u exists and admits a discrete mode expansion of the type:

$$v(\mathfrak{z}_1, \mathfrak{z}_2) \ \left(:= V_1(\mathfrak{z}_1 u, \mathfrak{z}_2 u) \right) = \sum_{n,\, m \in \mathbb{Z}} v_{nm}\, \mathfrak{z}_1^{-n-1} \mathfrak{z}_2^{-m-1} \,. \tag{27}$$

Here v_{nm} satisfy the following $\widehat{sp}(\infty)$ commutation relations:

$$\begin{aligned} [\, v_{n_1 m_1} \,,\, v_{n_2 m_2} \,] \ = \ & c\, n_1\, m_1 \left(\delta_{n_1,\,-n_2}\, \delta_{m_1,\,-m_2} + \delta_{n_1,\,-m_2}\, \delta_{m_1,\,-n_2} \right) + \\ & + n_1 \left(\delta_{n_1,\,-n_2}\, v_{m_1 m_2} + \delta_{n_1,\,-m_2}\, v_{m_1 n_2} \right) + \\ & + m_1 \left(\delta_{m_1,\,-n_2}\, v_{n_1 m_2} + \delta_{m_1,\,-m_2}\, v_{n_1 n_2} \right) . \end{aligned} \tag{28}$$

The vacuum representation of $\widehat{sp}(\infty)$ is characterized by the standard conditions

$$v_{nm}\, |0\rangle = 0 = \langle 0|\, v_{-n,\,-m} \quad \text{if} \quad n \geq 0 \quad \text{or} \quad m \geq 0 \,. \tag{29}$$

Consider the vector

$$\langle \Delta_n | = \frac{1}{n!}\, \langle 0| \begin{vmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \dots & \dots & \dots & \dots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{vmatrix} \tag{30}$$

which vanishes for V given by (25) and $n > N$. Its norm square is given by ([5] Lemma 2.5)

$$\| \, |\Delta_n\rangle \, \|^2 = \left\langle \Delta_n \middle| \Delta_n \right\rangle = (n+1)!\; c\, (c-1) \dots (c-n+1) \,. \tag{31}$$

It follows that a necessary and sufficient condition for the unitarity of the vacuum representation of $\widehat{sp}(\infty)$ is $c \in \mathbb{N}$ ([5] Theorem 5.1). For positive integer c, however, the $d = 2$ scalar field $\phi(x) = \frac{1}{2}\, V_1(x, x)$ can be presented according to (25) as a sum of normal products of free fields.

This somewhat disappointing result is based on the fact that the general GCI 4–point function of ϕ (for $d = 2$) is a multiple of the one for the normal square of a free field. It appears, therefore, promising to study the theory of a $d = 4$ GCI field for which, as we are going to demonstrate, this is not the case.

3 Lagrangean density. Expansion in twist 2 tensors of the bilocal field $V_1(x_1, x_2)$

A GCI model of a $d = 4$ scalar field $\mathcal{L}(x)$ in $4D$ Minkowski space is of particular interest since it may provide a non–perturbative description of a renormalization group fixed point of a non–abelian gauge theory (in which there is no gauge invariant field of dimension lower than 4).

We begin by writing down the solution of Eq. (21) satisfying the initial condition (22) and the symmetry relation (8) for $d = 4$ assuming there is no $d = 2$ scalar field in the theory (i. e. setting $a_0 = 0$):

$$f_4(s, t) = a_1 \left\{ \frac{(1-t)^2}{t} \left(1 - \frac{s}{1+t} \right) - \frac{2s}{1+t} \right\} + \\ + a_2 \left\{ \frac{(1-t)^4}{t^2} \left(1 - \frac{2\,s}{1+t} \right) - \frac{6\,s}{1+t} \, \frac{(1-t)^2}{t} + \frac{s^2}{t} \left(1 + \frac{(1-t)^2}{t} \right) \right\} . \tag{32}$$

The corresponding crossing symmetric contribution F_1 (5), (6) to the truncated 4–point function reads:

$$F_1(x_{ij}; d = 4) = (12)^2 (23)^2 (34)^2 (14)^2 \frac{1}{st} \sum_{\nu=1}^{2} a_\nu I_\nu(s, t) \; , \\ s^5 I_\nu \left(\frac{1}{s}, \frac{t}{s} \right) = I_\nu(s, t) = I_\nu(t, s) \; , \tag{33}$$

where I_ν are polynomials in the cross ratios:

$$I_1(s, t) = t \left[(1-t)^2 (1+t) - s \left(1 + t^2 \right) \right] + \\ + s \left[(1-s)^2 (1+s) - t \left(1 + s^2 \right) \right] + s\,t \left[(s-t)^2 (s+t) - s^2 - t^2 \right] , \\ I_2(s, t) = (1-t)^2 (1+t) \left(2 - 3\,t + 2\,t^2 \right) + (1-s)^2 (1+s) \left(2 - 3\,s + 2\,s^2 \right) - \\ - 2 + 4s\,t \left(s^2 + t^2 + 1 \right) - s\,t\,(s+t) \left(5\,s^2 - 8\,s\,t + 5\,t^2 \right) . \tag{34}$$

We shall demonstrate that the expression (6) for f_4 given by (32) is reproduced by an expansion of V_1 in conserved symmetric traceless tensors. In view of (13) and (18) this amounts to determining the (invariant under rescaling) structure constants $B_l\,(= N_l\, C_l^2\,(4))$ in such a way that

$$\sum_{l=1}^{\infty} B_l\,(1-t)^{2l}\, F\,(2l+1,\, 2l+1;\, 4l+2;\, 1-t) \; = \; = \left(1+\frac{1}{t}\right) \left\{ a_1\, \frac{(1-t)^2}{t} \; + \; a_2\, \frac{(1-t)^4}{t^2} \right\} . \tag{35}$$

We note that the system (35) is overdetermined: each B_l must satisfy two conditions to fit the coefficients to $(1-t)^{2l}$ and $(1-t)^{2l+1}$. Thus, the existence of a solution provides a non–trivial consistency check. One verifies that such a solution does exist and is given by

$$B_l \; = \; \frac{(2l)!\;(2l+1)!}{(4l-1)!} \left[a_1 + \frac{(2l+3)\,(l-1)}{2}\; a_2 \right] . \tag{36}$$

It is consistent with Wightman positivity for $a_1 \geq 0$, $a_2 \geq 0$.

The vanishing of $f_4\,(s,\,t)$ (32) – and hence of $\langle 0|\; V_1\,(x_1,\,x_2)\, V_1\,(x_3,\,x_4)\,|0\rangle =$ $(14)\,(23)\,(1+t)\; f_4\,(s,\,t)$ (6) – for $s=0,\; t=1$ (i.e. for $x_{34}=0$, or $x_{12}=0$ – according to (20)) implies

$$V_1\,(x_1,\,x_2) \; = \; x_{12}^{\mu}\, x_{12}^{\nu}\, T_{\mu\nu}\,(x_1,\,x_2) \tag{37}$$

where the limit $T_{\mu\nu}\,(x,\,x)$ exists and is a multiple of the stress–energy tensor (i.e., its 3–point function with V_1 satisfies the Ward–Takahashi identities).

We shall now demonstrate that a normal product $\mathcal{L}$ of free fields can only reproduce an expression of the type (32–36)) if it is a linear combination of free electromagnetic Lagrangeans

$$\mathcal{L}_0\,(x) \; = \; -\;\frac{1}{4}\;\; :F_{\mu\nu}\,(x)\, F^{\mu\nu}\,(x): \qquad \text{implying} \qquad a_1 \; = \; a_2\, . \tag{38}$$

In other words, only a 1–parameter subset of the 2–parameter family of expressions (36) belongs to the Borchers' class of free fields.

As a first step we shall verify the implication of (38) for a free Maxwell field F with 2–point function

$$\langle 0|\; F_{\mu_1\nu_1}\,(x_1)\, F_{\mu_2\nu_2}\,(x_2)\,|0\rangle = 4\, D_{\mu_1\nu_1\mu_2\nu_2}\,(x_{12}) = \Big\{ \partial_{\mu_1}\,(\partial_{\mu_2}\,\eta_{\nu_1\nu_2} - \partial_{\nu_2}\,\eta_{\nu_1\mu_2}) - - \,\partial_{\nu_1}\,(\partial_{\mu_2}\,\eta_{\mu_1\nu_2} - \partial_{\nu_2}\,\eta_{\mu_1\mu_2}) \Big\}\,(12)$$

$$\text{or} \qquad D_{\mu_1\mu_2\nu_1\nu_2}\,(x) \; = R_{\mu_1\mu_2}\,(x)\;\, R_{\nu_1\nu_2}\,(x) \; - \; R_{\mu_1\nu_2}\,(x)\;\, R_{\nu_1\mu_2}\,(x) \tag{39}$$

where

$$R^{\mu}{}_{\nu}(x) = \frac{r^{\mu}{}_{\nu}(x)}{4\pi^2 \rho_x} = (2\pi\rho_x)^{-2}(\rho_x \delta^{\mu}{}_{\nu} - 2x^{\mu}x_{\nu}), \quad \rho_x = x^2 + i\,0\,x^0 . \tag{40}$$

The OPE of the product of two $\mathcal{L}_0$'s then has the form

$$\begin{aligned} \mathcal{L}_0(x_1)\,\mathcal{L}_0(x_2) = \langle 12\rangle_0 + D_{\mu_1\nu_1\mu_2\nu_2}(x_{12}) :F^{\mu_1\nu_1}(x_1)\,F^{\mu_2\nu_2}(x_2): + \\ + :\mathcal{L}_0(x_1)\,\mathcal{L}_0(x_2): \end{aligned} \tag{41}$$

where

$$\begin{aligned} &\langle 12\rangle_0 = \frac{3}{(\pi\rho_{12})^4}, \\ &D_{\mu_1\nu_1\mu_2\nu_2}(x_{12}) :F^{\mu_1\nu_1}(x_1)\,F^{\mu_2\nu_2}(x_2): = \frac{8}{\pi^2\rho_{12}^3}\, V_1(x_1, x_2) \end{aligned} \tag{42}$$

for V_1 given by (37) with the bilocal tensor field

$$\begin{aligned} T(x_1, x_2; \zeta) :=\ & \zeta_{\mu}\, T^{\mu}{}_{\nu}(x_1, x_2)\, \zeta^{\nu} = \\ =\ & \frac{1}{4} :F^{\sigma\tau}(x_1)\,F_{\sigma\tau}(x_2):\zeta^2 - \zeta_{\mu} :F^{\sigma\mu}(x_1)\,F_{\sigma\nu}(x_2):\zeta^{\nu} . \end{aligned} \tag{43}$$

The 4–point function of T is computed from (39); setting $\zeta^{\,2} = 0 = \zeta'^{\,2}$ we find

$$\begin{aligned} 2^{-4}\,\langle 0|\, T(x_1, x_2; \zeta)\,T(x_3, x_4; \zeta')\,|0\rangle = \Big\{ R^{\sigma}{}_{\tau}(x_{14})\,(\zeta\cdot R(x_{14})\cdot\zeta') - \\ - (R^{\sigma}(x_{14})\cdot\zeta')\,(\zeta\cdot R_{\ \tau}(x_{14}))\Big\}\Big\{ R^{\tau}{}_{\sigma}(x_{23})\,(\zeta\cdot R(x_{23})\cdot\zeta') - \\ - (\zeta\cdot R_{\ \sigma}(x_{23}))\,(R^{\tau}(x_{23})\cdot\zeta')\Big\} + \{3 \leftrightarrow 4\} . \end{aligned} \tag{44}$$

The 4–point function of V_1 can be obtained from here by applying to the result the operator

$$\frac{1}{4}\left\{ \left(2\,x_{12}\cdot\frac{\partial}{\partial\zeta}\right)^2 - x_{12}^2\,\Box_{\zeta} \right\}\left\{ \left(2\,x_{34}\cdot\frac{\partial}{\partial\zeta'}\right)^2 - x_{34}^2\,\Box_{\zeta'} \right\} \tag{45}$$

thus recovering (6), (32) with $a_1 = a_2$, as stated.

Given that the expression (32) (the condition $a_0 = 0$) excludes the presence of a $d = 2$ scalar field in the OPE of type (41), we should like to exclude another conceivable candidate for a free theory in disguise given by a nonabelian ($SU(2)$) $F^a_{\mu\nu}$ that is a normal product of free longitudinal vector fields:

$$F^a_{\mu\nu}(x) = -g\,\epsilon^{abc}\,A^b_{\mu}(x)\;A^c_{\nu}(x) \qquad \text{for} \qquad \partial_{\mu}A^b_{\nu}(x) = \partial_{\nu}A^b_{\mu}(x) \tag{46}$$

$a,\, b,\, c \;=\; 1,\, 2,\, 3\,$. Here A^b_μ are (generalized) free fields with conformally invariant 2–point functions

$$\langle 0|\; A^a_\mu\,(x)\; A^b_\nu\,(x)\,|0\rangle \;=\; \frac{1}{2}\,\delta^{ab}\,R_{\mu\nu}\,(x_{12})\;, \tag{47}$$

ϵ^{abc} is the fully antisymmetric Levi–Civita tensor ($\epsilon^{123}=1$). Although the expression (46) is reminiscent to a pure gauge neither $F^a_{\mu\nu}$ nor the gauge invariant "Lagrangean"

$$\begin{aligned}\mathcal{L}_g\,(x) \;&=\; \frac{1}{4}\;\; :F^a_{\mu\nu}\,(x)\,F^{\mu\nu}_a\,(x)\!: \;=\\ &=\; \frac{g^2}{4}\;\; :\!\Big\{\big(A^a_\mu\,(x)\;A^\mu_a\,(x)\big)^2 - A^a_\mu\,(x)\;A^\mu_b\,(x)\;A^b_\nu\,(x)\;A^\nu_a\,(x)\Big\}\!:\end{aligned} \tag{48}$$

vanishes. A tedious but straightforward calculation gives the following leading term in the OPE of two $\mathcal{L}_g$:

$$\mathcal{L}_g\,(x_1)\;\mathcal{L}_g\,(x_2) \;=\; \frac{9}{4}\;g^4\;(12\,)^3\;A^a_\mu\,(x_1)\;r^{\mu\nu}\,(x_{12})\;A^a_\nu\,(x_2)\;+\;O\,(12)^2\;. \tag{49}$$

The first term in the right hand side gives rise to both the (standard) 2–point function of $\mathcal{L}$ and to a $V_1\,(x_1,\,x_2)$ for which the $d=2$ diagonal limit $V_1\,(x,\,x)$ is non–zero contrary to our assumption. Moreover, considering a general linear combination of the two $d=4$ scalars made out of a triplet of $d=1$ vector fields (appearing in the right hand side of (48)),

$$\mathcal{L}_{\xi\eta}\,(x) \;=\; \xi\,\big(A^a_\mu\,(x)\;A^\mu_a\,(x)\big)^2 \;-\; \eta\,\big(A^a_\mu\,(x)\;A^\mu_b\,(x)\;A^b_\nu\,(x)\;A^\nu_a\,(x)\big) \tag{50}$$

we deduce that the leading term in the OPE,

$$\mathcal{L}_{\xi\eta}\,(x_1)\;\mathcal{L}_{\xi\eta}\,(x_2) \approx 4\,\big(14\xi^2 - 16\,\xi\,\eta + 11\,\eta^2\big)\,(12\,)^3\,\big(A^a_\mu\,(x_1)\;r^{\mu\nu}\,(x_{12})\;A^a_\nu\,(x_2)\big)\,, \tag{51}$$

never vanishes for real $\xi,\,\eta\,$.

4 Twist 4 contribution. Concluding remarks

The general GCI and crossing symmetric truncated 4–point function of $\mathcal{L}\,(x)$ depends on 5 parameters ([5]). Three of them (a_ν) appear in the crossing symmetrized contribution F_1 of twist 2 fields. (We excluded one of them – setting $a_0 = 0$ – by demanding that no $d=2$ scalar field contributes to the OPE of two $\mathcal{L}$'s.) The remaining two parameters appear in the general expression for F_2 (defined by (5) for $\nu = 2$, $d = 4$):

$$F_2(x_{12}, x_{23}, x_{34}; 4) = (12)^2(23)^2(34)^2(14)^2 \left\{ b_1 \left(1 + s^2 + t^2\right) + b_2 \left(s + t + s\,t\right) \right\} \tag{52}$$

(the symmetrized contribution of twist 4 fields). For $2\,b_1 + b_2 \neq 0$ we deduce that the OPE of $V_2(x_1, x_2)$ involves $\mathcal{L}$, so that $\mathcal{L}$ may have a non–zero 3–point function.

To summarize: we have constructed a 4–parameter family of truncated 4–point functions $\mathcal{W}_4^t$ of a $d = 4$ scalar field $\mathcal{L}(x)$. Moreover, no lower dimensional fields contribute to $\mathcal{W}_4^t$. Only a 1–parameter subset of this 4–parameter family corresponds to a free (abelian gauge) field theory.

As a by-product we prove that the bilocal field V_1 appearing in the OPE (1) of two scalar fields is the sum of an infinite series of twist two conserved symmetric traceless tensors.

The result can be viewed as a first step to a non-perturbative construction of (gauge and) GCI invariant correlation function in a non-abelian gauge theory.

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